OPTIMAL CASUALTY INSURANCE AND REPAIR IN THE PRESENCE OF A SECURITY MARKET

AN CHEN[‡] AND PHILIP H. DYBVIG[*]

Abstract. We build a simple economic model of optimal casualty insurance based on a story about insuring a house. With endogenous repair and a security market that is complete over states distinguished by security payoffs, we have three main findings in our base model with additively-separable preferences. First, optimal repair depends on security market conditions, with full repair in inexpensive states and little or no repair in expensive states. Second, the optimal insurance payment equals the cost of optimal repair. Third, the agent is not made whole, since the loss is fully compensated only when damage is fully repaired. Weaker versions of the results hold more generally. Interestingly, when full repair is optimal it is fully insured.

Keywords: optimal casualty insurance, optimal repair
JEL-Code: G22

Introduction

In the simplest model of optimal insurance, there is actuarially fair insurance, and agents are fully insured in the sense that insurance makes them indifferent about whether a loss occurs or not. As discussed by Arrow (1963) and formalized more fully elsewhere in the literature,[1] there is less than full insurance in the presence of various frictions such as a cost of processing a claim and informational problems (such as adverse selection and moral

---

Date. October 7, 2010.

[‡] Department of Economics, University of Bonn, Adenauerallee 24-42, 53113 Bonn, Germany, Tel: 0049-228-739227, Fax: 0049-228-735048. E–Mail: an.chen@uni-bonn.de.

[*] Olin School of Business, Washington University in Saint Louis, Campus Box 1133 One Brookings Drive Saint Louis, MO 63130-4899, USA, Fax: 001-815-642-4318. E–Mail: phild@phildybvig.com.

An Chen acknowledges Bonn Graduate School of Economics for its supports and both of authors are grateful to Southwestern University of Finance and Economics IFS for their support.

[1]See for example Arrow (1971, 1973), Holmstrom (1979), Shavell (1979)), Rothschild and Stiglitz (1976), Towsend (1979), Rey (2003), and Doherty and Schlesinger (1983).

hazard). This paper gives an alternative reason why there may be less than full insurance, namely that if there is endogenous repair, there may be less than full insurance if it is not always optimal to undertake the repair. This idea is developed in a stylized example with a possible casualty loss on a house. The example has no aggregate actuarial risk and a frictionless security market for market-level risk. In our base model with additively-separable preferences, there is less than full insurance, but optimal repair is fully insured. Specifically, the optimal insurance policy fully insures the casualty loss in less expensive market states when repair is optimal, but does not pay any compensation in expensive market states in which repair is not optimal (even though there is not any friction or informational asymmetry).

We have three main results for additively-separable preferences. First, optimal repair depends on security market conditions, with full repair in inexpensive states and no repair in expensive states. Second, the optimal insurance payment equals the cost of optimal repair; therefore, there is no payment when there is no repair and the optimal repair is fully insured. Third, the agent is not made whole, since the loss is fully compensated only when it is fully repaired.

These results are also examined for more general preferences that may not be additively separable. It holds generally that the cost of full repair is fully insured. Except in the degenerate case of perfect substitutes, optimal repair in the examples still depends on the state of the security market, and optimal insurance does not make agents whole except when there is full repair. The case of perfect substitutes is different because housing and other consumption are in effect the same good.

For separable log utility and a Bernoulli distribution of loss we compare full insurance and no insurance constrained solutions with the optimal insurance solution. Optimal insurance obviously always delivers the highest value for the certainty equivalent. If repair is costly enough, no insurance is better than full insurance. Conversely if repair is sufficiently cheap, full insurance is better than no insurance. As the cost of repair increases, the optimal policy is to repair in fewer and fewer states, so there is less difference between optimal insurance and no insurance. However, the full insurance policy is worse and worse because there is more and more probability the insurance policy will pay for a repair that is not done. This just adds noise to financial wealth. As the cost of repair decreases, there are smaller and smaller differences between optimal insurance and no insurance, and between optimal insurance and full insurance, since the optimal, full, and no insurance payments are approximately zero.

The results of our paper have several empirical predictions. If preferences are additively separable, optimal insurance equals the cost of optimal repair. This predicts that insurance will pay off when there is repair but not when there is no repair. This is consistent with contracts that pay only in kind, for example, with cash going directly to the repair facility and not to the policy-holder. The extent to which there is repair when there is no payment is an interesting empirical question. If insurance pays for full repair and also pays something when there is no repair, it is consistent with non-additively-separable preferences. Probably the most striking prediction of the paper is that payments on insurance contracts depend on financial aggregates as well as the size and nature of the loss. This result does not seem to be consistent with existing insurance contracts and presents an interesting puzzle why this feature of our model is missing in practice.[2]

The remainder of the paper is organized as follows. Section 1 investigates optimal casualty insurance and repair problems in a stylized model where additively-separable preferences are assumed. Section 2 extends the base model to more general preferences. In Section 3, our benchmark problem (c.f. Section 1) is solved for the case of full and no insurance and comparisons are made between these cases and our optimal insurance framework. Section 4 concludes the paper.

1. A stylized model of optimal casualty insurance: base model

There are two points of time, 0 and 1. Each agent has endowment only at time 0: cash and a house of quality-adjusted size H purchased previously. At time 0, the agent may

[2] Our repair decision is different from the repair or replace decision in auto insurance (in which the insurance could declare the car a total loss). Presumably that decision does depend on market conditions in practice.

invest his endowment in securities in a market that is complete over states distinguished by security payoffs, and he may also purchase insurance through a mutual insurance company. Owning the house carries a risk of a casualty loss (this is what the insurance is for). The agent's casualty loss is given by the random variable C taking values in $[0, H)$. The objective probability distribution of the loss is common knowledge for all agents, and equals the true population average that will be realized. If $C = 0$, there is no casualty loss. We assume there is a positive probability of $C = 0$ conditional on ξ. C may be independent of ξ, but it does not have to be.

Investment in the market and buying insurance are both valued by a nonatomic state-price density[3] ξ with full support on $\Re_{++}$. The agent's budget constraint for terminal financial wealth P is

$$W_0 = E[\xi P(\xi, C)]. \tag{1}$$

P is the total terminal financial payment, which equals the final value of investments plus the insurance payment. Valuing a claim in a complete financial market using a state-price density (alternatively called a stochastic discount factor or pricing kernel) is standard in finance (see, e.g. Dybvig and Ross (1997)). Using the state-price density to price insurance claims as well implicitly assumes the private shocks underlying the claims are valued risk-neutrally. We offer two alternative motivations for this assumption leading to (1). One is a mutual insurance company's constraint for the representative customer when the insurance company does not face actuarial risk in aggregate. To see this, note that we can write

$$P(\xi, C) = E[P(\xi, C)|\xi] + (P(\xi, C) - E[P(\xi, C)|\xi]), \tag{2}$$

where $E[P(\xi, C)|\xi]$ is the market risk and $(P(\xi, C) - E[P(\xi, C)|\xi])$ is idiosyncratic risk which sums to zero across a continuum of agents.[4] Alternatively, it is the individual constraint given actuarially fair pricing of insurance against C conditional on ξ. This can

[3]We take the random state price density ξ to be exogenous, which means we are in partial equilibrium and is appropriate if we think of the set of agents insured by this company as being small compared to the economy. Nothing important would change if we endowed agents with financial wealth and determined ξ in a competitive equilibrium. Taking ξ nonatomic means the distribution of ξ has no mass point.

[4]We describe informally the diversification across a continuum of agents, but rigorous justification can be obtained by the construction in the important paper Green (1994).

be seen by the following equation resulting from the property of iterated expectation:

$$E[\xi E[P(\xi, C)|\xi]] = E[\xi P(\xi, C)]. \tag{3}$$

$E[P(\xi, C)|\xi]$ is the actuarially fair price given ξ, so the left-hand side of (3) gives the unconditional actuarially fair price of $P(\xi, C)$, which by the law of iterated expectations equals the valuation in (1) which equals the right-hand side of (2) based on the state price density.

We assume an additively-separable von Neumann-Morgenstern utility function $U_H(H)+U_W(P)$, where U_H is utility of housing and U_W is utility of wealth. All agents have identical preferences and identical initial wealth so that each agent has expected utility:

$$E[U_H(H - C + R(\xi, C)) + U_W(P(\xi, C) - \gamma R(\xi, C))].$$

We will assume U_H and U_W are C^1, strictly concave, and increasing. R is the value of the repair, and the cost of the repair is γR where $\gamma > 0$. Both P and R are functions of ξ and C, i.e. $P : \Re_{++} \times [0, H) \to \Re$ and $R : \Re_{++} \times [0, H) \to \Re$. We are assuming that the agents cannot sell their houses, perhaps because the total cost of selling the house and moving elsewhere is large. Therefore, the choice of whether to repair depends on the agent's preferences and not on some market valuation of whether the increase in house price is bigger than the cost of repair. It is reasonable to assume that R is no larger than the casualty loss, i.e. $0 \leq R \leq C$, that is, we are considering a repair, not an addition to the house ($R > C$) or selling off part of the house ($R < 0$).

In the solution, we take the usual approach to optimal contracting of computing the total optimal terminal wealth $P^*(\xi, C)$ (in the direct mechanism) and afterwards interpreting the optimal contract. We interpret the payment $W^*(\xi) = P^*(\xi, 0)$ as the payoff from investment in a security market, and we interpret the remainder $V^*(\xi, C) = P^*(\xi, C) - W^*(\xi)$ as payment on an insurance claim.[5]

[5] Alternatively, we could have another decomposition, e.g. if we do all investment through the insurance company and take $V^*(\xi, C) = P^*(\xi, C)$ and $W^*(\xi) = 0$. However, the decomposition in the text is sensible and gives interpretable results.

The optimal payment P and repair R solve the following problem:

$$\begin{aligned}
\text{Choose} \quad & P : \Re_{++} \times [0, H) \to \Re \text{ and } R : \Re_{++} \times [0, H) \to \Re \text{ to} && (4)\\
\text{maximize} \quad & E[U_H(H - C + R(\xi, C)) + U_W(P(\xi, C) - \gamma R(\xi, C))]\\
\text{subject to} \quad & W_0 = E[\xi P(\xi, C)] \text{ and}\\
\text{for all} \quad & \xi \in \Re_{++} \text{ and } C \in [0, H),\ 0 \le R(\xi, C) \le C.
\end{aligned}$$

Consider first the choice of $P(\xi, C)$ given $R(\xi, C)$. The Lagrangian function of our choice problem is

$$\begin{aligned}
\mathcal{L} =& E\left[U_H(H - C + R(\xi, C)) + U_W(P(\xi, C) - \gamma\, R(\xi, C))\right] + \lambda(W_0 - E\left[\xi P(\xi, C)\right])\\
=& E\left[U_H(H - C + R(\xi, C)) + U_W(P(\xi, C) - \gamma\, R(\xi, C)) + \lambda(W_0 - \xi P(\xi, C))\right] \qquad (5)
\end{aligned}$$

where λ is a Lagrangian multiplier. Maximizing the integrand of the Lagrangian function over the final wealth P leads to

$$U'_W(P(\xi, C) - \gamma\, R(\xi, C)) = \lambda\xi.$$

Letting I_W be the inverse function of U'_W, we obtain

$$P^*(\xi, C) = I_W(\lambda\, \xi) + \gamma\, R(\xi, C). \qquad (6)$$

When there is casualty loss, the optimal final wealth P^* depends on R, namely on the choice of whether to repair the house. Interestingly, the monetary impact of a casualty loss is endogenous because repair is endogenous and given additive separability insurance is for monetary risk. Since $R(\xi, 0) = 0$ (the only feasible choice), the first term in the right-hand side of (6) is the payoff W^* to investments, and the second term is the payoff $V^*(\xi, C)$ on the insurance claim. This form of the solution states that whatever the policy for repair (the R function), insurance will pay for all repairs. Therefore, the insurance will pay for the repair, if any, but pays nothing if there is a casualty loss that is not repaired, i.e. the cost of optimal repair is insured.

We will show that the optimal repair policy has full repair in inexpensive states (ξ small) and little or no repair in expensive states (ξ large). Maximizing the integrand of the Lagrangian (5) over R, we obtain

$$U'_H(H - C + R(\xi, C)) + U'_W(P(\xi, C) - \gamma R(\xi, C))(-\gamma) = 0$$

for an interior solution $0 < R < C$. At the optimum, substituting $P^*(\xi, C)$ from (6) into the above equation and imposing $0 \leq R \leq C$, we obtain the optimal repair policy

$$R^*(\xi, C) = \min\{\max\{I_H(\gamma\,\lambda\,\xi) - H + C, 0\}, C\}. \tag{7}$$

Obviously, the optimal repair policy is dependent on the state realization ξ. Let I_H be the inverse function of the marginal utility of housing U_H'. By the concavity of U_H, $I_H(\gamma\lambda\xi)$ is decreasing in ξ, which implies that the optimal repair $R^*(\xi, C)$ is non-increasing in ξ. Note that each of "repair all" $(R = C > 0)$ and "no repair" $(R = 0)$ and "partial repair" $(0 < R < C)$ is optimal for some ξ. We can rewrite (7) as[6]

$$R^*(\xi, C) = \begin{cases} 0 & \text{if } C = 0 \text{ or } \{C > 0 \text{ and } I_H(\gamma\,\lambda\,\xi) < H - C\} \\ I_H(\gamma\,\lambda\,\xi) - H + C & \text{if } C > 0 \text{ and } H - C < I_H(\gamma\,\lambda\,\xi) < H \\ C & \text{if } C > 0 \text{ and } I_H(\gamma\,\lambda\,\xi) > H. \end{cases}$$

Since λ is determined endogenously through the budget constraint, the exact optimal terminal wealth and optimal repair policy depend on the law for ξ, the utility function, the probability distribution of the loss, and the parameters H and W_0. The multiplier λ solves the budget constraint:[7]

$$W_0 = E[I_W(\lambda\xi)\,\xi] + E\left[\xi\gamma\left((I_H(\gamma\lambda\xi) - H + C)1_{\{H-C<I_H(\gamma\lambda\xi)<H\}} + C\,1_{\{I_H(\gamma\lambda\xi)>H\}}\right)\right]. \tag{8}$$

In the following, we take log utility as an example to analyze the optimal wealth, insurance and repair policy : $U_H(z) = \nu \log z$ and $U_W(z) = \log z$ and therefore $I_H(u) = \nu/u$ and $I_W(u) = 1/u$. We can write down the optimal repair policy (depending on ξ and C)

$$R^*(\xi, C) = \begin{cases} 0 & \text{if } C = 0 \quad \text{or} \quad \{C > 0 \text{ and } \xi > \frac{\nu}{\gamma\,\lambda\,(H-C)}\} \\ \frac{\nu}{\gamma\lambda\xi} + (C - H) & \text{if } C > 0 \text{ and } \frac{\nu}{\gamma\,\lambda\,H} < \xi < \frac{\nu}{\gamma\,\lambda\,(H-C)} \\ C & \text{if } C > 0 \text{ and } \xi < \frac{\nu}{\gamma\,\lambda\,H}. \end{cases} \tag{9}$$

[6]We are not concerned with knife-edge cases depending on ξ, because ξ is nonatomic and knife-edge cases have probability 0.

[7]Note that $H - C < I_H(\gamma\lambda\xi) < H$ implies $C > 0$.

State and casualty realization	$R^*(\xi, C)$	$W^*(\xi)$	$V^*(\xi, C)$
$C = 0$	0	$\frac{1}{\lambda\xi}$	0
$C > 0$ and $\xi > \frac{\nu}{\lambda(H-C)}$	0	$\frac{1}{\lambda\xi}$	0
$C > 0$ and $\frac{\nu}{\lambda H} < \xi < \frac{\nu}{\lambda(H-C)}$	$\frac{\nu}{\gamma\lambda\xi} + (C - H)$	$\frac{1}{\lambda\xi}$	$\frac{\nu}{\lambda\xi} + \gamma(C - H)$
$C > 0$ and $\xi < \frac{\nu}{\lambda H}$	C	$\frac{1}{\lambda\xi}$	γC

TABLE 1. Optimal repair policy R^*, portfolio payoff W^*, and insurance payoff V^* for additively-separable utilities with $U_H(z) = \nu \log z$ and $U_W(z) = \log z$ as a function of the casualty loss C and state price density ξ.

Consequently, the optimal terminal wealth has the following form:

$$P^*(\xi, C) = \begin{cases} \frac{1}{\lambda\xi} & \text{if } C = 0 \quad \text{or} \quad \{C > 0 \,\text{and}\, \xi > \frac{\nu}{\gamma\lambda(H-C)}\} \\ \frac{1}{\lambda\xi} + \frac{\nu}{\lambda\xi} + \gamma(C - H) & \text{if } C > 0 \text{ and } \frac{\nu}{\gamma\lambda H} < \xi < \frac{\nu}{\gamma\lambda(H-C)} \\ \frac{1}{\lambda\xi} + \gamma C & \text{if } C > 0 \text{ and } \xi < \frac{\nu}{\gamma\lambda H}. \end{cases} \tag{10}$$

The optimal final wealth can be decomposed into the optimal investment $W^*(\xi)$ and optimal insurance $V^*(\xi, C)$. As shown in Table 1, for given C the optimal insurance policy fully insures the casualty loss in very cheap states (and ξ is small), but does not pay off anything when the economy is down (and ξ is large). We have some additional states, i.e. moderately expensive states, full insurance is not optimal and instead a partial insurance is recommended. Note that optimal repair is fully insured. The multiplier λ is chosen to satisfy the budget constraint:

$$W_0 = E\left[\frac{1}{\lambda\xi}\xi\right] + E\left[\left(\frac{\nu}{\lambda} + \gamma(C - H)\xi\right) 1_{\{\frac{\nu}{\gamma\lambda H} < \xi < \frac{\nu}{\gamma\lambda(H-C)}\}} + \gamma C \xi\, 1_{\{\xi < \frac{\nu}{\lambda\gamma H}\}}\right], \tag{11}$$

which typically must be solved numerically, which is easy because (11) is decreasing in λ. As an example, assume C is Bernoulli, taking value $c > 0$ with probability p and 0 with probability $1 - p$. Assume further that the state price density process is lognormal with

$$\log\xi \sim N\left(-\left(r + \frac{1}{2}\eta^2\right), \eta^2\right), \tag{12}$$

where r is the continuously-compounded yield on a one-period bond and $\eta = \frac{\mu - r}{\sigma}$ is the market price of risk. The assumption is made as it would be in the Black-Scholes model with fixed coefficients and many other stationary models in continuous time with Vasicek term structure (see, for example, Section 3 of Dybvig (1988) or Examples 1 and

2 of Dybvig and Rogers (1997)). The assumption of lognormally distributed state price process ξ implies that (11) can be reformulated as:

$$
\begin{aligned}
W_0 =& \frac{1}{\lambda} + p\,\gamma\,c\,e^{-r} N\left(\frac{\log(\frac{\nu}{\lambda\gamma H}) + (r - \frac{1}{2}\eta^2)}{\eta}\right) \\
&+ p\,\gamma\,(c-H)e^{-r}\left(N\left(\frac{\log(\frac{\nu}{\lambda\gamma(H-c)}) + (r - \frac{1}{2}\eta^2)}{\eta}\right) - N\left(\frac{\log(\frac{\nu}{\lambda\gamma H}) + (r - \frac{1}{2}\eta^2)}{\eta}\right)\right) \\
&+ p\,\frac{\nu}{\lambda} N\left(\frac{\log(\frac{\nu}{\lambda\gamma(H-c)}) + (r + \frac{1}{2}\eta^2)}{\eta} - N\left(\frac{\log(\frac{\nu}{\lambda\gamma H}) + (r + \frac{1}{2}\eta^2)}{\eta}\right)\right) \qquad (13)
\end{aligned}
$$

where $N(x) := \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\frac{t^2}{2}} \mathrm{d}t$. The optimal value for λ can be determined numerically from the above equation. Equation (13) can be obtained from the Black-Scholes formula, and particularly the following two equalities resulting from the lognormal distribution of ξ are useful for the derivation:

$$
\begin{aligned}
E[\xi 1_{\{\xi<x\}}] =& e^{-r} N\left(\frac{\log x + (r - \frac{1}{2}\eta^2)}{\eta}\right) \\
E[1_{\{\xi<x\}}] =& N\left(\frac{\log x + (r + \frac{1}{2}\eta^2)}{\eta}\right).
\end{aligned}
$$

In the remainder of the section, we want to mention shortly what happens in the Bernoulli case if we assume partial repair is infeasible, so that the agent's repair policy R can be chosen to be 0 ("no repair") or c ("repair all"). If only these two repair policies are available, there exists a critical value ξ^* so that in less expensive states $\xi < \xi^*$, "repair all" is chosen over "no repair" and in more expensive states $\xi > \xi^*$, "no repair" is chosen. Let us now take a look at how ξ^* is determined. Given $C = c$, if R is chosen equal to c (corresponding to "repair all" case), it results in the value

$$
\mathcal{L}_{R=c} = U_H(H) + U_W\left(I_W(\lambda\,\xi)\right) + \lambda\left(W_0 - \xi\left(I_W(\lambda\,\xi) + \gamma c\right)\right),
$$

whereas if R is chosen equal to 0 (corresponding to "no repair" case), we have the value

$$
\mathcal{L}_{R=0} = U_H(H-c) + U_W\left(I_W(\lambda\,\xi)\right) + \lambda\left(W_0 - \xi I_W(\lambda\,\xi)\right).
$$

"Repair all" is better than "no repair" when

$$
\mathcal{L}_{R=c} > \mathcal{L}_{R=0} \quad \Leftrightarrow \quad \xi < \xi^* \equiv \frac{1}{\lambda\,\gamma\,c}(U_H(H) - U_H(H-c)). \qquad (14)
$$

Condition (14) indicates that in the less expensive states, "repair all" is preferable to "no repair", whereas in the more expensive states, "no repair" will be chosen. Because the insurance pays only if there is a repair, the optimal insurance policy fully insures the casualty loss in less expensive states and does not pay off anything in more expensive states.

Under additively-separable preferences, we have shown the following: a) Given C and H, optimal repair depends on the state of security market. "Repair all" is chosen when the economy is good (ξ small), and "no repair" is chosen when economy is bad (ξ large). If partial repair is feasible, then there is an intermediate range of states, in which partial repair is desirable. b) The optimal insurance payment is equal to optimal repair cost, which implies that there is no payment when there is no repair, but optimal repair is fully insured. c) Insurance does not make agents whole.

In the next section, the assumption of additively-separable preferences is relaxed to examine the robustness of the three main results.

2. Non-additively-separable preferences

In this section, we examine whether the results obtained in the base model hold under non-additively-separable preferences. We discuss several specific utilities and also provide some results for general preferences. Throughout we assume partial repair is feasible (as in our benchmark case in the previous section).

Under general (possibly non-additive) preferences, we find: a) Except in the degenerate case of perfect substitutes, optimal repair depends on the state of the security market. b) When full repair is optimal, it is fully insured, but otherwise the insurance payment may not equal the repair cost. c) In our example, insurance does not make agents whole except in the degenerate case of perfect substitutes.

Subsection 2.1 deals with the case in which housing and other consumption are perfect substitutes, i.e., are treated as a single good. Given perfect substitutes, the optimal insurance policy has full insurance and the agent is always made whole. In this case alone, optimal insurance and repair policy does not depend on the financial aggregates. Subsection 2.2 discusses the case in which housing and other consumption are perfect complements, i.e. utility is a function of the maximum of consumption and a multiple of housing. Given perfect complements, the optimal insurance has partial insurance and is not equal to the cost of repair. It holds that optimal insurance and optimal repair depend on the security market conditions. Subsections 2.3 and 2.4 analyze two cases in which we have the same static preferences for H and W as for log utilities but different risk preferences: $U(H,W) = -\frac{1}{HW}$ and $U(H,W) = H^{1/3}W^{1/3}$. In the former case, we obtain similar results as under perfect complements. In the latter case, we obtain a counterintuitive result. When there is a casualty loss and the agent chooses not to repair, the optimal insurance is negative. We interpret this as consistent with the very low relative risk aversion of these preferences. Subsection 2.5 provides some insight into a general non-additively-separable preferences, for which optimal full repair is fully insured.

2.1. **Perfect substitutes:** $U(a\,H + W)$**,** $a > 0$**.** Housing and wealth are perfect substitutes. In this case, the agent faces the following optimization problem:

$$\begin{aligned} \text{Choose} \quad & P : \Re_{++} \times [0,H) \to \Re \text{ and } R : \Re_{++} \times [0,H) \to \Re \text{ to} \\ \text{maximize} \quad & E[U(a(H - C + R(\xi,C)) + P(\xi,C) - \gamma R(\xi,C))] \\ \text{subject to} \quad & W_0 = E[\xi P(\xi,C)] \text{ and} \\ \text{for all} \quad & \xi \in \Re_{++},\ C \in [0,H),\ 0 \le R(\xi,C) \le C. \end{aligned} \tag{15}$$

We assume U is C^1, strictly concave, and increasing. The Lagrangian function is

$$\mathcal{L} = E\left[U(a(H - C + R(\xi,C)) + P(\xi,C) - \gamma R(\xi,C)) + \lambda(W_0 - \xi\, P(\xi,C))\right]$$

with λ being the Lagrangian multiplier. Maximizing the integrand of the Lagrangian function over the final wealth P leads to

$$U'(a(H - C + R(\xi,C)) + P(\xi,C) - \gamma R(\xi,C)) - \lambda\xi = 0.$$

Letting I be the inverse function of U', we can express the optimal terminal wealth as follows:

$$P^*(\xi,C) = I(\lambda\,\xi) - a\,H + (\gamma - a)R(\xi,C) + a\,C.$$

Substituting this back into the utility function, we obtain a utility of $U(I(\lambda\,\xi))$ which depends on ξ, but not on C. It implies that the optimal insurance policy is full insurance in the sense that given the state ξ of the market the insurance makes the agent indifferent about whether there is a casualty loss or not.

Given optimal $P^*(\xi, C)$, the derivative of the integrand of the Lagrangian function with respect to R is

$$
\begin{aligned}
& U'(a(H - C + R(\xi, C)) + P^*(\xi, C) - \gamma R(\xi, C))(a - \gamma) \\
=& (a-\gamma)\lambda\xi \begin{cases} < 0 & \text{if } a < \gamma \\ = 0 & \text{if } a = \gamma \\ > 0 & \text{if } a > \gamma. \end{cases}
\end{aligned}
$$

Therefore,

$$
R^*(\xi, C) \begin{cases} = 0 & \text{if } C = 0 \text{ or } a < \gamma \\ \in [0, C] & \text{if } C > 0 \text{ and } a = \gamma \\ = C & \text{if } C > 0 \text{ and } a > \gamma. \end{cases}
$$

In this case, optimal repair depends on the parameter a but not on the market state ξ.[8] If $C = 0$, the insurance pays off nothing and the optimal terminal wealth corresponds to $I(\lambda\xi) - aH$, and no repair is needed. If $a > \gamma$, full repair is always chosen, whereas if $a < \gamma$, no repair is always chosen. If $a = \gamma$, any feasible repair policy is optimal.

It is a quite intuitive result: since house and monetary wealth are perfect substitutes, a dollar's worth of repair costs γ dollars and always has a subjective benefit worth a dollar. If the benefit exceeds the cost ($a > \gamma$), repair is always optimal, whereas if cost exceeds the benefit ($\gamma > a$), repair is never optimal. Insurance follows from the traditional result for the single-good case absent frictions and information asymmetries, and always makes the agent whole whether or not repair is optimal. The only difference with repair is that when repair is preferred ($a > \gamma$), the possibility of repair reduces the cost of making the agent whole.

[8] If $a = 1$, the agent is indifferent between repairing and not repairing, and the agent could choose a repair policy that depends on the market state. However, this policy would be no better than choosing a policy that does not depend on the market state.

2.2. **Perfect complements.** Suppose that the agent's utility is characterized by $U(H, W) = \frac{(\min\{H, aW\})^{1-\rho}}{1-\rho}$. The interesting case has $\lambda > 0$, i.e. the budget constraint is strictly binding. The agent is satiated if all losses are repaired ($R(\xi, C) \equiv C$) and consumption is at least H/a. Therefore, if $W_0 \geq E[\xi(H/a + \gamma C)]$, then it is feasible to achieve the global optimum (conditional on H) and $\lambda = 0$. The choice problem under perfect complements is as follows:

$$
\begin{aligned}
\text{Choose} \quad & P : \Re_{++} \times [0, H) \to \Re \text{ and } R : \Re_{++} \times [0, H) \to \Re \text{ to} \\
\text{maximize} \quad & E\left[\frac{1}{1-\rho}(\min\{H - C + R(\xi, C), a(P(\xi, C) - \gamma R(\xi, C))\})^{1-\rho}\right] \\
\text{subject to} \quad & W_0 = E[\xi P(\xi, C)] \text{ and} \\
\text{for all} \quad & \xi \in \Re_{++}, \text{ and } C \in [0, H),\ 0 \leq R(\xi, C) \leq C.
\end{aligned}
\tag{16}
$$

Let $K(\xi, C) \equiv P(\xi, C) - \gamma R(\xi, C)$, which is consumption, i.e., the wealth after repair. It is useful to reparameterize the problem as a choice of $K(\xi, C)$ and $R(\xi, C)$ instead of $P(\xi, C)$ and $R(\xi, C)$. The minimum in the objective function of (16) implies we can infer a lot about the solution. For example, there is no use choosing $R > 0$ when $aK(\xi, C) < H - C$ and no use choosing $aK(\xi, C) > H$, since we would be paying a cost (positive since $\lambda > 0$) for no benefit. Also, when there is partial repair, we must have $H - C - R(\xi, C) = \alpha K(\xi, C)$ or else at least part of the cost of consumption or repair would be wasted. This implies w.l.o.g that $R(\xi, C) = \min\{C, \max\{0, aK(\xi, C) - (H - C)\}\}$. Substituting the definition of $K(\xi, C)$ and this expression for $R(\xi, C)$ into the integrand of the Lagrangian, we have that

$$
\mathcal{L} = \begin{cases}
\frac{(aK(\xi,C))^{1-\rho}}{1-\rho} + \lambda(W_0 - \xi K(\xi, C)) & \text{if } K < \frac{H-C}{a} \\
\frac{(aK(\xi,C))^{1-\rho}}{1-\rho} + \lambda(W_0 - \xi(K(\xi, C) + \gamma(aK(\xi, C) - (H - C)))) & \text{if } \frac{H-C}{a} < K < \frac{H}{a} \\
\frac{(H)^{1-\rho}}{1-\rho} + \lambda(W_0 - \xi(K(\xi, C) + \gamma C)) & \text{if } K > \frac{H}{a}
\end{cases}
$$

and the gradient is

$$
\frac{\partial \mathcal{L}}{\partial K} = \begin{cases}
a^{1-\rho}(K(\xi, C))^{-\rho} - \lambda\xi & \text{if } K < \frac{H-C}{a} \\
a^{1-\rho}(K(\xi, C))^{-\rho} - \lambda\xi(1 + \gamma a) & \text{if } \frac{H-C}{a} < K < \frac{H}{a} \\
-\lambda\xi & \text{if } K > \frac{H}{a}.
\end{cases}
$$

Letting $\frac{\partial \mathcal{L}}{\partial K} = 0$ leads to the optimal

$$K^*(\xi, C) = \begin{cases} (a^{\rho-1}\lambda\xi)^{-\frac{1}{\rho}} & K < \frac{H-C}{a} \\ (a^{\rho-1}\lambda\xi(1+\gamma a))^{-\frac{1}{\rho}} & \frac{H-C}{a} < K < \frac{H}{a} \\ \frac{H}{a} & K > \frac{H}{a}. \end{cases}$$

Note that the middle region is empty when $C = 0$. The Lagrangian is concave (strictly concave in the relevant region $K < H/a$) and not differentiable at the break points $(H - C)/a$ and H/a. At these points, the derivatives correspondence (subgradient) is the closed interval from the right derivative to the left derivative. The maximum is the value of K for which the derivative or an element of the subgradient is 0.
Substituting the optimal $K^*(\xi, C)$ back to the optimal repair policy, we have:

$$R^*(\xi, C) = \begin{cases} 0 & \text{if } C = 0 \text{ or } \left\{C > 0 \,\text{and}\, \xi > \frac{a(H-C)^{-\rho}}{(1+\lambda a)}\right\} \\ a(a^{\rho-1}\lambda\xi(1+\gamma a))^{-\frac{1}{\rho}} - (H - C) & \text{if } C > 0 \,\text{and}\, \frac{a(H-C)^{-\rho}}{(1+\lambda a)} > \xi > \frac{a(H)^{-\rho}}{(1+\lambda a)} \\ C & \text{if } C > 0 \,\text{and}\, \xi < \frac{a(H)^{-\rho}}{(1+\lambda a)}. \end{cases}$$

We obtain the optimal terminal wealth under perfect complements as follows:

$$P^*(\xi, C) = \begin{cases} (a^{\rho-1}\lambda\xi)^{-\frac{1}{\rho}} & \text{if } C = 0 \text{ or } \left\{C > 0 \,\text{and}\, \xi > \frac{a(H-C)^{-\rho}}{(1+\lambda a)}\right\} \\ (1+\gamma a)(a^{\rho-1}\lambda\xi(1+\gamma a))^{-\frac{1}{\rho}} - \gamma(H - C) & \text{if } C > 0 \,\text{and}\, \frac{a(H-C)^{-\rho}}{(1+\lambda a)} > \xi > \frac{aH^{-\rho}}{(1+\lambda a)} \\ (a^{\rho-1}\lambda\xi)^{-\frac{1}{\rho}} + \gamma C & \text{if } C > 0 \,\text{and}\, \xi < \frac{aH^{-\rho}}{(1+\lambda a)}. \end{cases}$$

The optimal terminal wealth can be decomposed into the optimal investment and optimal insurance as in Table 2. Under perfect complements, optimal repair still depends on security market conditions, with full repair in inexpensive states and little or no repair in expensive states. The agent is not made whole, since the loss is fully compensated only when it is fully repaired.

2.3. $U(H, W) = -\frac{1}{HW}$. In this case we have the same static preferences for H and W as for log utility, but different risk preferences. The integrand of the Lagrangian function (given ξ and C) is given by

$$\mathcal{L} = \frac{-1}{(H - C + R(\xi, C))(P(\xi, C) - \gamma\, R(\xi, C))} + \lambda(W_0 - \xi\, P(\xi, C))$$

State and casualty realization	$R^*(\xi, C)$	$W^*(\xi)$	$V^*(\xi, C)$
$C = 0$	0	$(a^{\rho-1}\lambda\xi)^{-\frac{1}{\rho}}$	0
$C > 0$ and $\xi > \frac{a(H-C)^{-\rho}}{(1+\lambda a)}$	0	$(a^{\rho-1}\lambda\xi)^{-\frac{1}{\rho}}$	0
$C > 0$ and $\frac{a(H-C)^{-\rho}}{(1+\lambda a)} > \xi > \frac{a(H)^{-\rho}}{(1+\lambda a)}$	$a(a^{\rho-1}\lambda\xi(1+\gamma a))^{-\frac{1}{\rho}}$ $-(H-C)$	$(a^{\rho-1}\lambda\xi)^{-\frac{1}{\rho}}$	$((1+\gamma a)^{1-\frac{1}{\rho}} - 1)(a^{\rho-1}\lambda\xi)^{-\frac{1}{\rho}}$ $-\gamma(H-C)$
$C > 0$ and $\xi < \frac{a(H)^{-\rho}}{(1+\lambda a)}$	C	$(a^{\rho-1}\lambda\xi)^{-\frac{1}{\rho}}$	γC

TABLE 2. Optimal repair policy R^*, portfolio payoff W^*, and insurance payoff V^* for perfect complements with $U(H, W) = \frac{(\min\{H, aW\})^{1-\rho}}{1-\rho}$ as a function of the casualty loss C and state price density ξ.

with λ being the Lagrangian multiplier. Maximizing over the integrand of the Lagrangian over $P(\xi, C)$, we obtain

$$\frac{1}{(H - C + R(\xi, C))(P(\xi, C) - \gamma\, R(\xi, C))^2} - \lambda\xi = 0. \tag{17}$$

Consequently, we have

$$P^*(\xi, C) = \left(\frac{1}{\lambda\,\xi}\frac{1}{H - C + R(\xi, C)}\right)^{\frac{1}{2}} + \gamma\, R(\xi, C). \tag{18}$$

Substituting this back into the utility function, we obtain a utility as a function of both ξ and C:

$$-\,(\lambda\,\xi)^{\frac{1}{2}}\,(H - C + R(\xi, C))^{-\frac{1}{2}}\,,$$

which depends on whether there is a casualty loss if there is not full repair. In other words, the agent is not made whole when there is a casualty loss and there is not full repair. In fact, for some ξ, partial insurance is desirable.

Given optimal $P^*(\xi, C)$, maximizing the integrand of the Lagrangian over R gives the first-order condition

$$\frac{(\lambda\xi)^{1/2}}{(H - C + R(\xi, C))^{3/2}} = \lambda\xi\gamma.$$

State and casualty realization	$R^*(\xi,C)$	$W^*(\xi)$	$V^*(\xi,C)$
$C=0$	0	$\left(\frac{1}{\lambda\xi H}\right)^{1/2}$	0
$C>0$ and $\xi>\frac{1}{\lambda\gamma^2}(H-C)^{-3}$	0	$\left(\frac{1}{\lambda\xi H}\right)^{1/2}$	$\left(\frac{1}{\lambda\xi(H-C)}\right)^{1/2}-\left(\frac{1}{\lambda\xi H}\right)^{1/2}$
$C>0$ and $\frac{1}{\lambda\gamma^2}H^{-3}<\xi<\frac{1}{\lambda\gamma^2}(H-C)^{-3}$	$\gamma^{-2/3}(\lambda\xi)^{-1/3}$ $-H+C$	$\left(\frac{1}{\lambda\xi H}\right)^{1/2}$	$2\gamma^{2/3}(\lambda\xi)^{-1/3}$ $-H+C-\left(\frac{1}{\lambda\xi H}\right)^{1/2}$
$C>0$ and $\xi<\frac{1}{\lambda\gamma^2}H^{-3}$	C	$\left(\frac{1}{\lambda\xi H}\right)^{1/2}$	γC

TABLE 3. Optimal repair policy R^*, portfolio payoff W^*, and insurance payoff V^* for $U(H,W)=-\frac{1}{HW}$ as a function of the casualty loss C and state price density ξ.

Taking account of the constraint for R, we obtain the following optimal repair policy

$$R^*(\xi,C)=\min\left\{\max\left\{\gamma^{-2/3}(\lambda\xi)^{-1/3}-H+C,0\right\},C\right\}$$
$$=\begin{cases}0 & \text{if } C=0 \text{ or } \left\{C>0 \text{ and } \xi>\frac{1}{\lambda\gamma^2}(H-C)^{-3}\right\}\\ \gamma^{-2/3}(\lambda\xi)^{-1/3}-H+C & \text{if } C>0 \text{ and } \frac{1}{\lambda\gamma^2}H^{-3}<\xi<\frac{1}{\lambda\gamma^2}(H-C)^{-3}\\ C & \text{if } C>0 \text{ and } \xi<\frac{1}{\lambda\gamma^2}H^{-3}.\end{cases}$$

Substituting the optimal repair policy back to the optimal terminal wealth leads to

$$P^*(\xi,C)=\begin{cases}\left(\frac{1}{\lambda\xi(H-C)}\right)^{1/2} & \text{if } C=0 \text{ or } \left\{C>0 \text{ and } \xi>\frac{1}{\lambda\gamma^2}(H-C)^{-3}\right\}\\ 2\gamma^{2/3}(\lambda\xi)^{-1/3}-H+C & \text{if } C>0 \text{ and } \frac{1}{\lambda\gamma^2}H^{-3}<\xi<\frac{1}{\lambda\gamma^2}(H-C)^{-3}\\ \left(\frac{1}{\lambda\xi H}\right)^{1/2}+\gamma C & \text{if } C>0 \text{ and } \xi<\frac{1}{\lambda\gamma^2}H^{-3}.\end{cases}$$

In Table 3 the optimal terminal wealth is decomposed to two parts: optimal investment $W^*(\xi)$ in the financial market and optimal insurance $V^*(\xi,C)$. It is observed that partial insurance might be desirable, depending on the optimal repair policy which is further dependent on the realization of ξ and C. It does not hold that optimal repair is equal to optimal insurance, but it is true that full repair is fully insured.

2.4. $U(H,W)=H^{1/3}W^{1/3}$**.** Again the static preferences are equivalent to log utility, but relative risk aversion over wealth is less than 1. The Lagrangian function can be expressed as

$$\mathcal{L} = E\left[(H - C + R(\xi, C))^{\frac{1}{3}}(P(\xi, C) - \gamma R(\xi, C))^{\frac{1}{3}} + \lambda(W_0 - \xi\, P(\xi, C))\right].$$

Maximization the integrand of the Lagrangian over P leads to

$$\frac{1}{3}(H - C + R(\xi, C))^{\frac{1}{3}}(P(\xi, C) - \gamma R(\xi, C))^{-\frac{2}{3}} - \lambda\xi = 0. \tag{19}$$

As a result, we have

$$P^*(\xi, C) = \frac{(H - C + R(\xi, C))^{1/2}}{(3\lambda\,\xi)^{3/2}} + \gamma\, R(\xi, C). \tag{20}$$

Plugging this back in the utility function, we have a utility of

$$\left(\frac{H - C + R(\xi, C)}{3\,\lambda\,\xi}\right)^{\frac{1}{2}},$$

which depends on ξ and casualty loss C. It means that full insurance is not an optimal insurance policy. The agent's utility is not equalized in all circumstances, particularly for the two scenarios: if there is no casualty loss; and if there is casualty loss and "no repair" is chosen. If there is casualty loss but no repair, the agent is not made whole.

Concerning the optimal repair policy, given optimal $P^*(\xi, C)$, maximizing the integrand of the Lagrangian over R, we have

$$\begin{aligned}&\frac{1}{3}(H - C + R(\xi, C))^{-2/3}(P^*(\xi, C) - \gamma R(\xi, C))^{-2/3}(-(H - C)\gamma + P^*(\xi, C) - 2\gamma R(\xi, C))\\ =& -\gamma\lambda\xi + \frac{1}{3}\frac{(H - C + R(\xi, C))^{-1/2}}{(3\lambda\xi)^{1/2}}.\end{aligned}$$

Hereby we have used (19) and (20). Taking account of the constraints for R, we obtain the following optimal repair policy:

$$\begin{aligned}R^*(\xi, C) =& \min\left\{\max\left\{\gamma^{-2}(3\lambda\xi)^{-3} - H + C, 0\right\}, C\right\}\\ =& \begin{cases} 0 & \text{if } C = 0 \text{ or } \left\{C > 0 \text{ and } \xi > \frac{1}{3\lambda}\left(\frac{1}{(H-C)\gamma^2}\right)^{\frac{1}{3}}\right\} \\ \gamma^{-2}(3\lambda\xi)^{-3} - H + C & \text{if } C > 0 \text{ and } \frac{1}{3\lambda}\left(\frac{1}{H\gamma^2}\right)^{\frac{1}{3}} < \xi < \frac{1}{3\lambda}\left(\frac{1}{(H-C)\gamma^2}\right)^{\frac{1}{3}} \\ C & \text{if } C > 0 \text{ and } \xi < \frac{1}{3\lambda}\left(\frac{1}{H\gamma^2}\right)^{\frac{1}{3}}. \end{cases}\end{aligned}$$

State and casualty realization	$R^*(\xi, C)$	$W^*(\xi)$	$V^*(\xi, C)$
$C = 0$	0	$\frac{H^{1/2}}{(3\lambda\xi)^{3/2}}$	0
$C > 0$ and $\xi > \frac{1}{\lambda\gamma^2}(H - C)^{-3}$	0	$\frac{H^{1/2}}{(3\lambda\xi)^{3/2}}$	$\frac{(H-C)^{1/2}}{(3\lambda\xi)^{3/2}} - \frac{(H)^{1/2}}{(3\lambda\xi)^{3/2}} < 0$
$C > 0$ and $\frac{1}{\lambda\gamma^2}H^{-3} < \xi < \frac{1}{\lambda\gamma^2}(H - C)^{-3}$	$\gamma^{-2}(3\lambda\xi)^{-3} - H + C$	$\frac{H^{1/2}}{(3\lambda\xi)^{3/2}}$	$2\gamma^{-1}(3\lambda\xi)^{-3} - \gamma(H - C) - \frac{(H)^{1/2}}{(3\lambda\xi)^{3/2}}$
$C > 0$ and $\xi < \frac{1}{\lambda\gamma^2}H^{-3}$	C	$\frac{H^{1/2}}{(3\lambda\xi)^{3/2}}$	γC

TABLE 4. Optimal repair policy R^*, portfolio payoff W^*, and insurance payoff V^* for $H^{1/3}W^{1/3}$ as a function of the casualty loss C and state price density ξ.

Substituting the optimal repair policy back in $P^*(\xi, C)$, we obtain

$$P^*(\xi, C) = = \begin{cases} \frac{(H-C)^{1/2}}{(3\lambda\xi)^{3/2}} & \text{if } C = 0 \text{ or } \left\{C > 0 \text{ and } \xi > \frac{1}{\lambda\gamma^2}(H - C)^{-3}\right\} \\ 2\gamma^{-1}(3\lambda\xi)^{-3} - \gamma(H - C) & \text{if } C > 0 \text{ and } \frac{1}{\lambda\gamma^2}H^{-3} < \xi < \frac{1}{\lambda\gamma^2}(H - C)^{-3} \\ \frac{(H)^{1/2}}{(3\lambda\xi)^{3/2}} + \gamma C & \text{if } C > 0 \text{ and } \xi < \frac{1}{\lambda\gamma^2}H^{-3}. \end{cases}$$

In Table 4 the optimal terminal wealth is split again into two parts: optimal investment $W^*(\xi)$ from the securities market and optimal insurance $V^*(\xi, C)$. Very interestingly, we obtain a quite counterintuitive result. If the agent chooses not to repair at all when there is a casualty, the optimal insurance is negative, i.e. the insurance is not going to pay off anything for the casualty and furthermore ask the agent to provide it a certain amount. It is an anti-insurance scenario and results from the fact that the utility here leads to an extremely low relative relation aversion coefficient (RRA < 1). So there is a reporting issue, i.e. unlike in all the other cases we consider, the agent actually has an incentive to hide the loss.

2.5. **General utility.** Some of the results are available for a general (not necessarily additively-separable) utility function $U(H, W)$, which is assumed to be twice differentiable, strictly monotone and strictly concave. Specifically, $U_i > 0$, $i = 1, 2$ and (U_{ii}) is negative definite, i.e. $U_{11} < 0$, $U_{22} < 0$ and $U_{11}U_{22} - (U_{12})^2 < 0$. Under these assumptions, the optimal payment P and repair R solve the following problem:

$$\begin{aligned} \text{Choose} \quad & P : \Re_{++} \times [0, H) \to \Re \text{ and } R : \Re_{++} \times [0, H) \to \Re \text{ to} \\ \text{maximize} \quad & E[U(H - C + R(\xi, C), P(\xi, C) - \gamma R(\xi, C))] \\ \text{subject to} \quad & W_0 = E[\xi P(\xi, C)] \text{ and} \\ \text{for all} \quad & \xi \in \Re_{++}, \text{ and } C \in [0, H), \ 0 \le R(\xi, C) \le C. \end{aligned} \tag{21}$$

Since there is no feasible repair necessary when there is no casualty loss, for $C = 0$, the associated Lagrangian function is

$$\mathcal{L}_{C=0} = E\Big[U(H, P(\xi, 0)) + \lambda(W_0 - \xi P(\xi, 0))\Big].$$

Maximizing over the integrand of the Lagrangian leads to

$$U_2(H, P(\xi, 0)) = \lambda\xi. \tag{22}$$

For $C > 0$, the associated Lagrangian function

$$\begin{aligned} \mathcal{L}_{C>0} =& E\Big[U(H - C + R(\xi, C), P(\xi, C) - \gamma\, R(\xi, C)) + \lambda(W_0 - \xi P(\xi, C)) \\ & + \mu_1 R(\xi, C) + \mu_2(C - R(\xi, C))\Big] \end{aligned} \tag{23}$$

is maximized over $R(\xi, C) \in [0, C]$. The solution to the optimal terminal wealth and the optimal repair policy is characterized as follows:

$$\begin{aligned} & U_2(H - C + R(\xi, C), P(\xi, C) - \gamma\, R(\xi, C)) = \lambda\xi, \\ & U_1(H - C + R(\xi, C), P(\xi, C) - \gamma\, R(\xi, C)) \\ =& \gamma U_2\Big(H - C + R(\xi, C), P(\xi, C) - \gamma\, R(\xi, C)\Big) + \mu_2 - \mu_1, \end{aligned}$$

where $\mu_1 \ge 0$, $\mu_2 \ge 0$, $\mu_1 R(\xi, C) = 0$ and $\mu_2(C - R(\xi, C)) = 0$. As seen in the specific examples, the optimal repair policy can have full repair, partial repair or no repair.

In the following, let us examine whether optimal repair is fully insured, i.e. whether

$$V(\xi, C) = P(\xi, C) - P(\xi, 0) = \gamma R(\xi, C). \tag{24}$$

Note that

$$\lambda\xi = U_2(H - C + R(\xi, C), P(\xi, C) - \gamma R(\xi, C)) = U_2(H, P(\xi, 0)). \tag{25}$$

For $R(\xi, C) = C$ (full repair),

$$U_2(H, P(\xi, C) - \gamma C) = U_2(H, P(\xi, 0)). \tag{26}$$

Since U is assumed to be strictly monotone and strictly concave, (26) leads to

$$V(\xi, C) = \gamma R(\xi, C) = \gamma C = P(\xi, C) - P(\xi, 0).$$

It indicates that full repair is always fully insured. When there is partial repair, let us first assume that (24) holds. Under this assumption, (25) can be rewritten to:

$$\lambda \xi = U_2(H - C + R(\xi, C), P(\xi, 0)) = U_2(H, P(\xi, 0)). \tag{27}$$

Certainly, (27) does not hold generally. It does hold under additively-separable preferences, where U_2 does not depend on the first argument.[9] Therefore, under the general utility, optimal repair is not always fully insured, but full repair is fully insured.

Another interesting question is whether the agent is made whole. In particular, we want to examine whether the agent is made full when there is not full repair. In other words, do we obtain the same utility in these two cases: "$C = 0$" and "$C > 0$, $R = 0$"? From the first order conditions, we know that

$$U_2(H, P(\xi, 0)) = U_2(H - C, P(\xi, C)) = \lambda \xi.$$

Does it imply

$$U(H, P(\xi, 0)) = U(H - C, P(\xi, C))? \tag{28}$$

The equality in (28) does not hold in general, but it is true for perfect substitutes (analyzed earlier) and we cannot rule out that it can happen by accident.

3. Full and no insurance

This section aims to examine the degree of efficiency of our optimal insurance and repair framework. We solve our base problem (c.f. Section 1) now under full insurance and no insurance. We then compare optimal, full and no insurance by looking at the certainty

[9]There is probably a sense in which this is necessary as well if (27) holds for a rich enough set of problems, since $f_2(x, y)$ is independent of x for all y iff f is additively separable.

equivalents achieved under these three cases.

3.1. **Full insurance.** In case of full insurance, insurance pays for any casualty loss, independent of whether repair is optimal. Hence, the optimal payment from the financial market W and repair R solve the following problem:

$$\begin{aligned} \text{Choose} \quad & W : \Re_{++} \to \Re \text{ and } R : \Re_{++} \times [0, H) \to \Re \text{ to} \\ \text{maximize} \quad & E[U_H(H - C + R(\xi, C)) + U_W(W(\xi) + \gamma C - \gamma R(\xi, C))] \\ \text{subject to} \quad & W_0 = E[\xi W(\xi) + \gamma C] \text{ and} \\ \text{for all} \quad & \xi \in \Re_{++} \text{ and } C \in [0, H),\ 0 \le R(\xi, C) \le C. \end{aligned} \tag{29}$$

Assume that the casualty loss is Bernoulli, taking value $c > 0$ with probability p and 0 with probability $1 - p$. The Lagrangian function of our choice problem is

$$\begin{aligned} \mathcal{L} =& p\, E\left[U_H(H - c + R(\xi, c)) + U_W(W(\xi) + \gamma c - \gamma\, R(\xi, c))\right] \\ & + (1-p)\, E\left[U_H(H) + U_W(W(\xi))\right] + \lambda(W_0 - E\left[\xi(W(\xi) + p\gamma c)\right]) \\ =& p\, E\left[U_H(H - c + R(\xi, c)) + U_W(W(\xi) + \gamma c - \gamma\, R(\xi, c))\right] \\ & + (1-p)\, E\left[U_H(H) + U_W(W(\xi))\right] + \lambda(W_0 - p\gamma c e^{-r} - E[\xi\, W(\xi)]), \end{aligned} \tag{30}$$

where λ is a Lagrangian multiplier. Given ξ, maximizing the integrand of the Lagrangian function over the final wealth W leads to

$$p\, U_W'(W(\xi) + \gamma c - \gamma\, R(\xi, c)) + (1-p) U_W'(W(\xi)) = \lambda \xi. \tag{31}$$

The optimal terminal wealth $W(\xi)$ depends on the repair policy. Note that (31) holds whether or not R has a boundary solution. If R has an interior solution, we have additionally the following first order condition of the Lagrangian with respect to R:

$$p\,(U_H'(H - c + R(\xi, c)) + U_W'(W(\xi) + \gamma c - \gamma R(\xi, c))(-\gamma)) = 0. \tag{32}$$

Taking log utilities as an example, $U_W(z) = \log z$ and $U_H(z) = \nu \log z$, we can reformulate (31) and (32) to

$$\frac{p}{W(\xi) + \gamma c - \gamma\, R(\xi, c)} + \frac{1-p}{W(\xi)} = \lambda \xi \tag{33}$$

$$\frac{\nu}{H - c + R(\xi, c)} - \frac{\gamma}{W(\xi) + \gamma c - \gamma R(\xi, c)} = 0. \tag{34}$$

If $R(\xi, c)$ takes the corner solution, i.e. for $R(\xi, c) = 0$, we solve (33) and obtain

$$W^{FI}(\xi) = \frac{1 - \lambda\xi\gamma c + \sqrt{(1 - \lambda\xi\gamma c)^2 + 4\lambda\xi(1-p)\gamma c}}{2\lambda\xi}.$$

Solving (33) for $R(\xi, c) = c$, we have

$$W^{FI}(\xi) = \frac{1}{\lambda\xi}.$$

If R has an interior solution, the optimal wealth and repair are determined by solving the equation system (33) and (34):

$$W^{FI}(\xi) = \frac{(1 + p\nu - H\gamma\lambda\xi) + \sqrt{(1 + p\nu - H\gamma\lambda\xi)^2 + 4(1-p)H\gamma\lambda\xi}}{2\lambda\xi} \tag{35}$$

$$R^{FI}(\xi, c) = \frac{\nu}{(1+\nu)\gamma} W^{FI}(\xi) + \frac{\nu c - (H - c)}{1 + \nu}. \tag{36}$$

To sum up, the optimal terminal payment from the financial market and the optimal repair policy under full insurance are as follows:

$$W^{FI}(\xi) = \begin{cases} \frac{1}{\lambda\xi} & \text{if } \xi < \xi_1^* \\ \frac{(1+p\nu - H\gamma\lambda\xi) + \sqrt{(1+p\nu - H\gamma\lambda\xi)^2 + 4(1-p)H\gamma\lambda\xi}}{2\lambda\xi} & \text{if } \xi_1^* < \xi < \xi_2^* \\ \frac{1 - \lambda\xi\gamma c + \sqrt{(1-\lambda\xi\gamma c)^2 + 4\lambda\xi(1-p)\gamma c}}{2\lambda\xi} & \text{if } \xi > \xi_2^* \end{cases}$$

$$R^{FI}(\xi, c) = \begin{cases} c & \text{if } \xi < \xi_1^* \\ \frac{\nu}{(1+\nu)\gamma} \frac{(1+p\nu - H\gamma\lambda\xi) + \sqrt{(1+p\nu - H\gamma\lambda\xi)^2 + 4(1-p)H\gamma\lambda\xi}}{2\lambda\xi} + \frac{\nu c - (H-c)}{1+\nu} & \text{if } \xi_1^* < \xi < \xi_2^* \\ 0 & \text{if } \xi > \xi_2^* \end{cases}$$

$$\text{with} \quad \xi_1^* = \frac{\nu}{H\gamma\lambda}, \text{ and } \xi_2^* = \frac{(1-p)H + \frac{H-c-\nu c}{\nu}(1 + p\nu)}{\lambda\gamma\left(\left(\frac{H-c-\nu c}{\nu}\right)^2 + \frac{(H-c-\nu c)}{\nu} H\right)}.$$

The Lagrangian multiplier λ is determined by solving the budget constraint, i.e.

$$E[\xi W^{FI}(\xi)] + p\gamma c e^{-r} = W_0.$$

Or more specifically, λ is computed by numerically solving

$$\begin{aligned} W_0 - p\gamma c e^{-r} = E\Bigg[&\frac{1}{\lambda} 1_{\{\xi < \xi_1^*\}} + \frac{1 - \lambda\xi\gamma c + \sqrt{(1 - \lambda\xi\gamma c)^2 + 4\lambda\xi(1-p)\gamma c}}{2\lambda} 1_{\{\xi > \xi_2^*\}} \\ &+ \frac{(1 + p\nu - H\gamma\lambda\xi) + \sqrt{(1 + p\nu - H\gamma\lambda\xi)^2 + 4(1-p)H\gamma\lambda\xi}}{2\lambda} 1_{\{\xi_1^* < \xi < \xi_2^*\}} \Bigg]. \end{aligned}$$

The expected utility is given by

$$
\begin{aligned}
EU =&(1-p)E[U_H(H)+U_W(W^{FI}(\xi))]+pE[U_H(H-c+R^{FI}(\xi,c)) \\
&+U_W(W^{FI}(\xi)+\gamma c-\gamma R^{FI}(\xi,c))]
\end{aligned}
$$

and the certainty equivalent is defined by $I_W(EU-U_H(H))$.

3.2. **No insurance.** In case of no insurance, the optimal payment from the financial market W and repair R solve the following problem:

$$
\begin{aligned}
\text{Choose}\quad & W:\Re_{++}\to\Re \text{ and } R:\Re_{++}\times[0,H)\to\Re \text{ to} \\
\text{maximize}\quad & E[U_H(H-C+R(\xi,C))+U_W(W(\xi)-\gamma R(\xi,C))] \\
\text{subject to}\quad & W_0=E[\xi W(\xi)] \text{ and} \\
\text{for all}\quad & \xi\in\Re_{++} \text{ and } C\in[0,H),\ 0\le R(\xi,C)\le C.
\end{aligned}
\tag{37}
$$

Assume that the casualty loss is Bernoulli. The Lagrangian function of our choice problem is

$$
\begin{aligned}
\mathcal{L} =&p\,E\left[U_H(H-c+R(\xi,c))+U_W(W(\xi)-\gamma\,R(\xi,c))\right]+(1-p)\,E\left[U_H(H)+U_W(W(\xi))\right] \\
&+\lambda(W_0-E\left[\xi W(\xi)\right]),
\end{aligned}
$$

where λ is a Lagrangian multiplier. Given ξ, maximizing the integrand of the Lagrangian function over the final wealth W and R (if R has interior solutions) leads to

$$
\begin{aligned}
p\,U_W'(W(\xi)-\gamma\,R(\xi,c))+(1-p)U_W'(W(\xi)) =&\lambda\xi \\
p\,(U_H'(H-c+R(\xi,c))+U_W'(W(\xi)-\gamma R(\xi,c))(-\gamma)) =&0.
\end{aligned}
$$

Under log utilities, $U_W(z)=\log z$ and $U_H(z)=\nu\log z$, the above two equalities become

$$
\begin{aligned}
\frac{p}{W(\xi)-\gamma\,R(\xi,c)}+\frac{1-p}{W(\xi)} =&\lambda\xi \\
\frac{\nu}{H-c+R(\xi,c)}-\frac{\gamma}{W(\xi)-\gamma R(\xi,c)} =&0.
\end{aligned}
$$

Solving the equation system, we obtain the optimal terminal payment from the financial market and the optimal repair policy as follows:

$$
\begin{aligned}
W^{NI}(\xi) =& \frac{(1+p\nu-(H-c)\gamma\lambda\xi)+\sqrt{(1+p\nu-(H-c)\gamma\lambda\xi)^2+4(1-p)(H-c)\gamma\lambda\xi}}{2\lambda\xi} \\
R^{NI}(\xi,c) =& \frac{\nu}{(1+\nu)\gamma}W^{NI}(\xi)-\frac{H-c}{1+\nu}.
\end{aligned}
$$

If $R(\xi,c)$ takes the corner solution $R(\xi,c)=0$, we have

$$
W^{NI}(\xi)=\frac{1}{\lambda\xi},
$$

and the corner solution $R(\xi,c)=c$,

$$
W^{NI}(\xi)=\frac{1+\lambda\xi\gamma c+\sqrt{(1+\lambda\xi\gamma c)^2-4\lambda\xi(1-p)\gamma c}}{2\lambda\xi}.
$$

Under no insurance, the optimal terminal wealth from the financial market and the optimal repair policy are given by

$$
W^{NI}(\xi)=\begin{cases}
\frac{1+\lambda\xi\gamma c+\sqrt{(1+\lambda\xi\gamma c)^2-4\lambda\xi(1-p)\gamma c}}{2\lambda\xi} & \text{if } \xi<\xi_1^* \\
\frac{(1+p\nu-(H-c)\gamma\lambda\xi)+\sqrt{(1+p\nu-(H-c)\gamma\lambda\xi)^2+4(1-p)(H-c)\gamma\lambda\xi}}{2\lambda\xi} & \text{if } \xi_1^*<\xi<\xi_2^* \\
\frac{1}{\lambda\xi} & \text{if } \xi>\xi_2^*
\end{cases}
$$

$$
R^{NI}(\xi,c)=\begin{cases}
c & \text{if } \xi<\xi_1^* \\
\frac{\nu}{(1+\nu)\gamma}\frac{(1+p\nu-(H-c)\gamma\lambda\xi)+\sqrt{(1+p\nu-(H-c)\gamma\lambda\xi)^2+4(1-p)(H-c)\gamma\lambda\xi}}{2\lambda\xi}-\frac{H-c}{1+\nu} & \text{if } \xi_1^*<\xi<\xi_2^* \\
0 & \text{if } \xi>\xi_2^*
\end{cases}
$$

$$
\text{with}\quad \xi_1^* = \frac{(1-p)(H-c)\gamma+\frac{\gamma(H+\nu c)}{\nu}(1+p\nu)}{\lambda\left(\left(\frac{\gamma(H+\nu c)}{\nu}\right)^2+\frac{\gamma(H+\nu c)}{\nu}(H-c)\gamma\right)},\ \text{and}\ \xi_2^*=\frac{\nu}{(H-c)\gamma\lambda}.
$$

The Lagrangian multiplier is determined by solving the budget constraint, i.e.

$$
W_0 = E[\xi W^{NI}(\xi)].
$$

Like in the full insurance case, we need to rely on numerical techniques to determine the Lagrangian multiplier solving the following equation:

$$\begin{aligned}W_0 =&E\left[\frac{1+\lambda\xi\gamma c+\sqrt{(1+\lambda\xi\gamma c)^2-4\lambda\xi(1-p)\gamma c}}{2\lambda}1_{\{\xi<\xi_1^*\}}+\frac{1}{\lambda}1_{\{\xi>\xi_2^*\}}\right.\\&\left.+\frac{(1+p\nu-(H-c)\gamma\lambda\xi)+\sqrt{(1+p\nu-(H-c)\gamma\lambda\xi)^2+4(1-p)(H-c)\gamma\lambda\xi}}{2\lambda}1_{\{\xi_1^*<\xi<\xi_2^*\}}\right].\end{aligned}$$

The expected utility is given by

$$\begin{aligned}EU =&(1-p)E[U_H(H)+U_W(W^{NI}(\xi))]+pE[U_H(H-c+R^{FI}(\xi,c))\\&+U_W(W^{FI}(\xi)-\gamma R^{FI}(\xi,c))]\end{aligned}$$

and the certainty equivalent is defined by $I_W(EU-U_H(H))$.

3.3. **Efficiency Comparison.** In what follows, we want to compare our optimal insurance framework with full and no insurance by considering the certainty equivalent (CE) defined by

$$I_W(EU-U_H(H)),$$

where the expected utility EU differs in each case and depends on the corresponding optimal wealth and repair: $(P^*(\xi,C),R^*(\xi,C))$ in optimal insurance, $\left(W^{FI}(\xi),\ R^{FI}(\xi,C)\right)$ in full insurance, and $\left(W^{NI}(\xi),\ R^{NI}(\xi,C)\right)$ in no insurance. We use CE_{OI}, CE_{FI} and CE_{NI} to denote the certainty equivalent achieved under optimal, full and no insurance. In Table 5, certainty equivalents are computed for varying γ for log utilities ($U_H(z)=\nu\log z$ and $U_W(z)=\log z$). There are several observations. First, as γ increases, CE_{OI}, CE_{FI} and CE_{NI} all decrease. A higher γ leads to a higher cost of repair, which is bad news and this is strict (if not numerically significant), since there are always some states with ξ small enough in which repair is optimal for all the policies. Second, we observe that $CE_{OI}>CE_{FI}$ and $CE_{OI}>CE_{NI}$, because CE_{OI} is determined under optimal insurance policy and the other strategies are not, although the suboptimality may not be numerically significant. Third, the value of CE_{NI} is higher than CE_{FI} when γ is large enough (here $\gamma=1.5$, 2.5, 5) and smaller than CE_{FI} when γ is small enough (here $\gamma=0.1$, 0.5, 1). It is due to the fact that as γ increases, the difference $CE_{OI}-CE_{NI}$ goes to zero because optimal policy is to repair in fewer and fewer states. In our example, CE_{OI} and CE_{NI}

are very close for $\gamma = 2.5$ and $\gamma = 5$. However, the full insurance policy is worse and worse because of buying more and more insurance that actually increases risks and in fact the certainty equivalent approaches 0 and then the problem becomes infeasible because the cost of required insurance exhausts wealth and there is nothing left for consumption. Fourth, as γ decreases, the difference $CE_{OI} - CE_{NI}$ approaches 0 and so will the difference $CE_{OI} - CE_{FI}$ behave. For $\gamma = 0.1$, the differences are already very close to zero. The reason is that the cost of repair is going to zero and we repair almost all of the time almost for free. However, we also have

$$\frac{CE_{OI} - CE_{FI}}{CE_{OI} - CE_{NI}} \to 0$$

because the optimal policy has repair almost all the time (insured because preferences are additively separable), so that full insurance is approximately optimal.

It seems that there are no substantial differences between the certainty equivalents in Table 5 and in particular the loss is uniformly quite small if we choose between full or no insurance (the choice in practice) instead of buying optimal insurance. This is just an example, but perhaps this is the solution to the puzzle of why contracts in practice are not contingent on the state of the economy: it just does not matter much.[10] For a one-year time horizon, the dispersion of ξ is not big enough to matter much. Hence, we have added Table 6 where μ, σ and r are set for a 10-year time horizon. In this case, the dispersion of states prices is much greater. This makes full insurance particularly unattractive because it forces us to buy insurance we do not want exactly when it is very expensive, so that no insurance is better than full insurance except in the cheapest case.

4. Conclusion

We study optimal insurance and repair of casualty loss in the presence of a security market. We show that under additively-separable preferences, optimal repair depends on security market conditions; the optimal insurance payment equals the cost of optimal repair; and the agent is not made whole. Under more general (possibly non-additively-separable preferences), optimal repair still depends on the state of the security market and

[10]It may be tempting to say this *is* the solution to the puzzle, but the example has low risk aversion (log $\Rightarrow$ RRA $= 1$, a single period, and exogenous house ownership. It would be difficult to build a compelling estimate of the true magnitude of the utility loss.

γ	CE_{OI} (optimal insurance)	CE_{FI} (full insurance)	CE_{NI} (no insurance)
0.1000	108.1394	108.1394	108.1295
0.5000	104.9726	104.9704	104.7218
1.0000	101.6656	101.3647	101.3106
1.5000	100.2058	98.8705	100.0585
2.5000	99.6718	95.3916	99.6608
5.0000	99.6394	84.8034	99.6394

TABLE 5. Comparison of certainty equivalents of optimal insurance, full insurance, no insurance with parameters: $W_0 = 100$, $H = 100$, $p = 0.25$, $c = 30$, $r = 0.03$, $\mu = 0.08$, $\sigma = 0.15$, $\nu = 1$.

γ	CE_{OI} (optimal insurance)	CE_{FI} (full insurance)	CE_{NI} (no insurance)
0.1000	233.9678	233.9646	233.9396
0.5000	229.5315	229.1640	229.2949
1.0000	225.8291	224.1149	225.4749
1.5000	223.4063	219.6464	223.0269
2.5000	220.5172	211.3195	220.1713
5.0000	217.5574	189.9811	217.3360

TABLE 6. Comparison of certainty equivalents of optimal insurance, full insurance, no insurance with parameters: $W_0 = 100$, $H = 100$, $p = 0.25$, $c = 30$, $r = 0.03 \cdot 10$, $\mu = 0.08 \cdot 10$, $\sigma = 0.15 \cdot \sqrt{10}$, $\nu = 1$.

insurance does not make agents whole (except in the degenerate case of perfect substitutes). Interestingly, when full repair is optimal it is fully insured.

The model predicts that under additively-separable preferences insurance contracts will only pay off on optimal repairs and do not provide any compensation for damage that is not to be repaired, and indeed some insurance contracts will pay for repairs but not provide cash directly. To the extent that insurance does pay off even when repairs are not made this might be evidence of non-separable preferences such as those analyzed in Section 2. It is an interesting question whether contracts that pay off in cash do so knowing that the beneficiary has a strong incentive to make a repair, or whether there is a useful function served by paying off in cash that is economically different than if the insurer paid for the repairs directly. For example, in a setting with asymmetric information, the second-best contract may pay off when there are no repairs because the insurance company does not have enough information to know whether repairs are optimal. In this case, the efficiency gain from allowing the beneficiary to opt out of the repair may dominate any incentive effects of giving the beneficiary an option.

One striking feature of the model is that the optimal insurance payoff depends both on the size of the loss and also the state of the economy. Since insurance contracts typically do not depend on economic aggregates, this presents a puzzle. It would be useful to come up with a model and/or empirical analysis to resolve this puzzle. A possible answer is given in Section 3 because this is a small loss in certainty equivalent from having full or no insurance instead of optimal insurance. However, this example is extraordinarily stylized and not very persuasive.

Modifying the model with assumptions appropriate for specific insurance contracts might lead to new insights and sharper predictions. More generally, the optimal contracting framework in the paper could be used to analyze other situations, such as the choice of an optimal mortgage, in which agents face both public and private risk.

References

Arrow, Kenneth J. 1963. "Uncertainty and the Welfare Economics of Medical Care." *American Economic Review*, 53: 941–973.

Arrow, Kenneth J. 1971. "On Optimal Insurance Policies." *Appendix to Essay 8 in "Essays in the Theory of Risk Bearing"*, Chicago, Markham.

Arrow, Kenneth J. 1973. "Optimal Insurance and Generalized Deductibles." *Rand Report.*

Doherty, Neil A. and Harris Schlesinger. 1983. "Optimal Insurance in Incomplete Markets", *Journal of Political Economy*, 91: 1045–1054.

Dybvig, Philip H. 1988. "Inefficient Dynamic Portfolio Strategies, or, How to Throw Away a Million Dollars in the Stock Market." *Reviews of Financial Studies*, 1: 67–88.

Dybvig, Philip H. and L.C.G. Rogers. 1997. "Recovery of Preferences from Observed Wealth in a Single Realization", *Reviews of Financial Studies*, 10: 151–174.

Dybvig, Philip H. and Stephen A. Ross. 1997. "Arbitrage." *a contribution to The New Palgrave: a Dictionary of Economics 1, New York: Stockton Press*, 1987: 100–106.

Green, Edward J. 1994. "Individual-level Randomness in a Nonatomic Population." *Minnesota Economics and Minnesota Fed Working Paper, available at http://129.3.20.41/eps/ge/papers/9402/9402001.pdf.*

Holmstrom, Bengt. 1979. "Moral Hazard and Observability." *The Bell Journal of Economics,* 10(2): 74–91.

Rey, Beatrice. 2003. "A Note on Optimal Insurance in the Presence of a Nonpecuniary Background Risk." *Risk Theory and Decision*, 54: 73–83.

Rotschild, Michael. and Joseph E. Stiglitz. 1976. "Equilibrium in Competitive Insurance Markets: An Essay on the Economics of Imperfect Information." *Quarterly Journal of Economics*, 90: 639–649.

Shavell, Steven. 1979. "On Moral Hazard and Insurance." *The Quarterly Journal of Economics*, 93(4) (Nov., 1979): 541–562.

Townsend, Robert. 1979. "Optimal Contracts and Competitive Markets with Costly State Verification." *Journal of Economic Theory*, 22: 265–293.